The club of simplicial sets

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Abstract

A club structure is defined on the category of simplicial sets. This club generalizes the operad of associative rings by adding "amalgamated" products.

1 Introduction

There is a straightforward way to define operads in the monoidal category $(Cat, \times)$: just apply the standard definition. However, since Cat is not just a category, but a 2-category, such a definition is of very limitied value. For one thing, the action of symmetric groups through functors would not be the correct action in most applications. Instead, one would need symmetric groups to act by morphisms.

In this paper we make use of another way, the 2-categorical structure of Cat makes itself felt. In the definition of operads one parameterizes the procedure of taking several points in a set and composing them into one point. Of course sets can be substituted with objects in any other symmetric monoidal category, but the principle remains the same: we compose strings of elements.

Let $M \in Cat$ be a category. A string of objects $\{A_1, \ldots, A_n\} \subseteq M$ is the same as a diagram $\mathbf{n} \to M$, where $\mathbf{n}$ is a discrete category on n-objects. Here of course we can take any diagram $D \to M$, where D is not necessarily discrete, and try to "compose" it. This kind of compositions cannot be described in terms of operads. We need the notion of a club instead.

In their full generality clubs were developed by G.M.Kelly in [KG74], and they are the way to encode associativity of compositions when we com-

pose arbitrary diagrams, and not just the ones parameterized by discrete categories.

Recall that the main axiom of operads is associativity of compositions of operations. Clubs provide a formalization for the same axiom, but in the more general case, where operations can have arbitrary diagrams as inputs, and not just strings. Of course, operads are clubs of a particular kind.

In this paper we define a club structure on the category of simplicial sets $SSet$. This club generalizes the operad of associative rings by adding compositions of elements *relative* to other elements.

Discrete simplicial sets give just an associative product, non-discrete simplicial sets give "amalgamated" associative products. An example of such amalgamated products are monoidal globular categories in [Ba98].

Here is the structure of the paper: in section 2 we recall the definition of clubs. The general definition, given in [KG74], takes place in an arbitrary 2-category. We do not need this generality, and we consider only clubs in Cat. As an example we show that set-theoretic operads are clubs in Cat of a particular kind.

We also use a different notation from [KG74]. A club in Cat is a monoid in the category of diagrams in Cat. This category has a very important monoidal product, which is a straightforward generalization to categories of the semidirect product of groups. Therefore we use the symbol $\ltimes$ to designate this monoidal structure.

For any diagram $\underline{D}$ in Cat, the functor $\underline{D} \ltimes - : Cat \to Cat$ is an instance of what is called *a familial* 2*-functor* ([WM07], [SR00]).

In section 3, starting from the category $SSet$ of simplicial sets, we define the structure of a club on $\underline{SSet}$, where $\underline{SSet}$ is a diagram in Cat, parameterized by $SSet$, with every simplicial set mapped to its category of simplices.

In fact, we define two clubs: one on the entire category of simplicial sets, and another on the subcategory, consisting of injective morphisms. The latter is important in applications, when we have a category M, with an $\underline{SSet}$-algebra structure on it, and we want to have an $\underline{SSet}$-algebra structure on the subcategory of mono-morphisms in M.

A note on notation: When working with sets we use the approach of universes ([SGA4]), in particular we speak of *a* small category Set of sets, meaning sets in a given universe. Consequently we have *a* small category $SSet$ of simplicial sets.

2 Semi-direct product and clubs in Cat

Definition 1 *Let $\underline{Cat}$ be the following category:*

- *An object $\underline{D} \in \underline{Cat}$ is a pair $\{D, R\}$, where D is a small category, and $R : D \to Cat$ is a functor.*
- *A morphism $\underline{F} : \underline{D} \to \underline{D}'$ in $\underline{Cat}$ is a pair $\{F, \rho\}$ where $F : D \to D'$ is a functor, and ρ is a natural transformation, making the following diagram commutative:*

$$\begin{tikzcd} D \arrow[rr, "F"] \arrow[dr, "R"'] & {} \arrow[d, phantom, "\xleftarrow{\rho}"] & D' \arrow[dl, "R'"] \\ & Cat & \end{tikzcd}$$

Now we are going to define a monoidal structure $\{\ltimes, \underline{\mathbf{1}}\}$ on $\underline{Cat}$. We start with products of objects.

Let $\underline{D}, \underline{D}' \in \underline{Cat}$ be objects, let $d \in D$ be an object, and let $R(d) \overset{\psi}{\to} D'$ be a functor. We define a category $R(d) \ltimes_\psi \underline{D}'$ as follows:

- Objects of $R(d) \ltimes_\psi \underline{D}'$ are pairs $\{a, b\}$, where $a \in R(d)$ and $b \in R'\psi(a)$.
- Morphisms of $R(d) \ltimes_\psi \underline{D}'$ are pairs $\{\alpha, \beta\}$, where $\alpha : a_1 \to a_2$ is a morphism in $R(d)$, and $\beta : R'\psi(\alpha)(b_1) \to b_2$ is a morphism in $R'\psi(a_2)$.
- Composition of $\{a_1, b_1\} \xrightarrow{\{\alpha_1,\beta_1\}} \{a_2, b_2\} \xrightarrow{\{\alpha_2,\beta_2\}} \{a_3, b_3\}$ is

$$\{\alpha_2\alpha_1, \beta_2 R'\psi(\alpha_2)(\beta_1)\}.$$

It is easy to check that $R(d) \ltimes_\psi \underline{D}'$ is indeed a category, and similarity between this construction and semi-direct product of groups is obvious.

Different from the case of groups, we can put together all $R(d) \ltimes_\psi \underline{D}'$'s for all $d \in D$ and all $\psi : R(d) \to D'$ to get a diagram $\underline{D} \ltimes \underline{D}' \in \underline{Cat}$.

First we describe the parameterizing category. Let $D \ltimes D'$ be the small category, defined as follows:

- Objects of $D \ltimes D'$ are pairs $\{d, \psi_d\}$, where $d \in D$ is an object, and $\psi_d : R(d) \to D'$ is a functor.

- Morphisms in $D \ltimes D'$ are pairs $\{f, \phi\}$, where $f : d_1 \to d_2$ is a morphism in D, and ϕ is a natural transformation, making the following diagram commutative:

$$\begin{tikzcd}
R(d_1) \arrow[rr, "R(f)"] \arrow[dr, "\psi_{d_1}"'] \arrow[rr, phantom, shift right=6ex, "\xrightarrow{\phi}"] & & R(d_2) \arrow[dl, "\psi_{d_2}"] \\
& D' &
\end{tikzcd}$$

Now we define $R \ltimes R' : D \ltimes D' \to Cat$. As we said above, we would like to collect all $R(d) \ltimes_{\psi_d} \underline{D'}$'s into one diagram, so on objects $R \ltimes R'$ is clear:

$$R \ltimes R' : \{d, \psi_d\} \longmapsto R(d) \ltimes_{\psi_d} \underline{D'}.$$

It is straightforward then to define the action of $R \ltimes R'$ on morphisms of $D \ltimes D'$ by composing functors and natural transformations in an obvious way. Here is the explicit description: for a morphism $\{d_1, \psi_{d_1}\} \overset{\{f,\phi\}}{\to} \{d_2, \psi_{d_2}\}$ we define the functor

$$R \ltimes R'(\{f, \phi\}) : R(d_1) \ltimes_{\psi_{d_1}} \underline{D'} \longrightarrow R(d_2) \ltimes_{\psi_{d_2}} \underline{D'}$$

as follows: let $\{\alpha, \beta\} : \{a_1, b_1\} \to \{a_2, b_2\}$ be a morphism in $R(d_1) \ltimes_{\psi_{d_1}} \underline{D'}$, then we define $R \ltimes R'(\{f, \phi\})(\{\alpha, \beta\})$ to be

$$\{R(f)(a_1), R'(\phi_{a_1})(b_1)\} \xrightarrow{\{R(f)(\alpha), R'(\phi_{a_2})(\beta)\}} \{R(f)(a_2), R'(\phi_{a_2})(b_2)\}$$

Functoriality of this construction is obvious, since all we do here is composing functors and natural transformations.

Definition 2 *Let $\underline{D}, \underline{D'} \in \underline{Cat}$ be objects. We define their* **semi-direct product** *$\underline{D} \ltimes \underline{D'}$ to be $\{D \ltimes D', R \ltimes R'\} \in \underline{Cat}$ as above.*

Now we describe the unit objects for $\ltimes$. Let $\mathbf{1}$ be a discrete category on one object, and let $\underline{\mathbf{1}} \in \underline{Cat}$ consist of $\mathbf{1}$, mapped to itself in Cat. It is easy to see that for any $\underline{D} \in \underline{Cat}$ we have canonically

$$\underline{D} \ltimes \underline{\mathbf{1}} \cong \underline{D} \cong \underline{\mathbf{1}} \ltimes \underline{D}.$$

We do not prove the following proposition, since it is a consequence of the general result, proved in [KG74].

Proposition 1 *Let $\underline{Cat}$ be the category of small diagrams in Cat (Definition 1). The semi-direct product $\ltimes$ (Definition 2) together with $\underline{\mathbf{1}}$ define a monoidal structure on $\underline{Cat}$.*

We would like to note that $\ltimes$ is *not* symmetric. This is easy to see from the following simple example: let $D = \mathbf{1}$, $D' = \mathbf{2}$ (discrete categories on one and two objects respectively); let $R : D \to Cat$ be defined by by mapping the only object to $\mathbf{2} \in Cat$, and let $R' : D' \to Cat$ be defined by mapping every object to $\mathbf{1} \in Cat$. Then

$$\underline{D} \ltimes \underline{D'} \ncong \underline{D'} \ltimes \underline{D}.$$

Definition 3 **A club** *in Cat is a monoid in $(\underline{Cat}, \ltimes, \underline{\mathbf{1}})$.*

As with most monoids, we will be interested in modules over a club in Cat. Given a club $\underline{C}$, it is straightforward to define a $\underline{C}$-module in $\underline{Cat}$, but in practice we would like clubs to act on categories, i.e. objects of Cat, rather than $\underline{Cat}$. For that we need a bit of notation.

Let D be a small category. There are is a natural way to associate an object in $\underline{Cat}$ to D. Let $\mathbb{D} := D \mapsto \mathbf{1}$ be the diagram in Cat, having D as the parameterizing category, s.t. every object in D is mapped to $\mathbf{1} \in Cat$.

Notice that the assignment $D \mapsto \mathbb{D}$ is a functor from Cat to $\underline{Cat}$, and it is left adjoint to the forgetful functor $\underline{Cat} \to Cat$, that maps every diagram to its parameterizing category.

Definition 4 *Let $\underline{C}$ be a club in Cat, and let $M \in Cat$ be a category. We define $\underline{C}(M)$ to be the parameterizing category of $\underline{C} \ltimes \mathbb{M}$.*

A $\underline{C}$-algebra *is a category M, together with a functor $\underline{C}(M) \to M$, satisfying the usual associativity conditions.*

Now we are ready to consider examples.

1. Let P be a set-theoretic non-symmetric operad, i.e. we have a sequence of sets $\{P_n\}_{n\geq 0}$, a chosen element $e \in P_1$, and a sequence of compositions

   $$\gamma_{m_1,\ldots,m_n} : P_n \times P_{m_1} \times \ldots \times P_{m_n} \to P_{m_1+\ldots+m_n},$$

   satisfying the usual conditions of associativity and unitality.

   Now we construct a diagram in Cat, starting with P, and for every $n > 0$ a choice of a discrete category $\mathbf{n}$ having an ordered set of n

objects. The parameterizing category P is the discrete category having $\coprod_{n\geq 0} P_n$ as the set of objects. Every $p \in P_n \subseteq P$ is mapped to $\mathbf{n}$. We will denote the resulting diagram by $\underline{P}$.

This construction works for any $\mathbb{N}$-collection in Set, in particular for $P \circ P$, where

$$(P \circ P)_k = \coprod_{m_1+\ldots+m_n=k} P_n \times P_{m_1} \times \ldots \times P_{m_n}.$$

Proof of the following proposition is straightforward.

Proposition 2 *1. For any collection P in Set we have*

$$\underline{P \circ P} \cong \underline{P} \ltimes \underline{P}. \tag{1}$$

2. The correspondence (1) defines a bijection between the set of operadic compositions $\{\gamma_{m_1,\ldots,m_n}\}$ on P, and the set of $\ltimes$-monoidal structures

$$\{F, \rho\} : \underline{P} \ltimes \underline{P} \to \underline{P}, \qquad \{I, \iota\} : \underline{\mathbf{1}} \to \underline{P},$$

s.t. ρ, ι are natural equivalences, preserving the order on $\mathbf{n}$'s.

Let S be a set, and suppose P acts on it. Then it is easy to see how to translate such an action into the structure of a $\underline{P}$-algebra on S, considered as a discrete category.

2. Now let P be an operad in Set. Here, in addition to choosing a discrete category $\mathbf{n}$ on n objects, for each $n \geq 1$ we fix an isomorphism

$$\mathbb{S}_n \cong Aut(\mathbf{n}).$$

Then we can define a diagram $\underline{P} \in \underline{Cat}$ as follows: the parameterizing category P has $\coprod_{n\geq 0} P_n$ as the set of objects, and $\forall p, q \in P_n \subseteq P$, we put $Hom(p, q)$ to be the set of all $\sigma_n \in \mathbb{S}_n$, s.t. $\sigma_n(p) = q$; the functor $R : P \to Cat$ maps every $p \in P_n \subseteq P$ to $\mathbf{n}$, and every morphism $p \xrightarrow{\sigma} q$ to the corresponding endofunctor on $\mathbf{n}$.

It is clear that $\underline{P}$ is indeed an object in $\underline{Cat}$, and we can apply the same technique to every Σ-collection in Set. Different from the non-symmetric case, we have that in general $\underline{P} \ltimes \underline{P} \ncong \underline{P \circ P}$. However, we have a natural inclusion $\underline{P} \ltimes \underline{P} \to \underline{P \circ P}$, and hence we can conclude the following.

Proposition 3 *For any Σ-collection P in Set, there is a bijection between operadic structures on P, and $\ltimes$-monoidal structures*

$$\{F, \rho\} : \underline{P} \ltimes \underline{P} \to \underline{P}, \qquad \{I, \iota\} : \mathbf{\underline{1}} \to \underline{P},$$

s.t. ρ, ι are natural equivalences, that preserve order on $\mathbf{n}$'s.

Proof: The only difference here from the non-symmetric case is the action of symmetric groups. Since $P_n \times P_{m_1} \times \ldots \times P_{m_n}$ carries the action of only $\mathbb{S}_n \times \mathbb{S}_{m_1} \times \ldots \times \mathbb{S}_{m_n}$, and hence in general it is not an $\mathbb{S}_{m_1+\ldots+m_n}$-set, we have that $\underline{P} \ltimes \underline{P} \ncong \underline{P \circ P}$, and we cannot proceed as in Proposition 2.

In defining operads one extends $P_n \times P_{m_1} \times \ldots \times P_{m_n}$ by tensoring it with $\mathbb{S}_{m_1+\ldots+m_n}$ over $\mathbb{S}_n \times \mathbb{S}_{m_1} \times \ldots \times \mathbb{S}_{m_n}$. However, while this makes definition of an operad cleaner, it is not really needed, and it is enough to postulate equivariance only with respect to $\mathbb{S}_n \times \mathbb{S}_{m_1} \times \ldots \times \mathbb{S}_{m_n}$. ■

Also here it is easy to see how to translate the notion of a P-algebra in *Set* into a $\underline{P}$-algebra in *Cat*.

3 The club of simplicial sets

In the previous section we have considered two examples of $\ltimes$-monoids of a special kind. In general a $\ltimes$-monoid is given by an object $\underline{D}$ in $\underline{Cat}$, together with morphisms

$$\{F, \rho\} : \underline{D} \ltimes \underline{D} \to \underline{D}, \qquad \{I, \iota\} : \mathbf{\underline{1}} \to \underline{D}$$

in $\underline{Cat}$, satisfying the usual associativity and unit axioms.

In the case of set-theoretic operads we have required that ρ and ι are not just natural transformations, but natural *equivalences.* This requirement was a consequence of the way we represented operads: all operadic compositions were encoded in the parameterizing category P, i.e. operations are represented as objects in P. The functor $P \to Cat$ was there only to keep track of the arity of these operations.

Now we consider a case where ρ is not required to be invertible. This case is the main example for a "diagrammatic" operadic action on categories: here we do not compose strings of objects, but diagrams of objects, and hence categories in the image of $P \to Cat$ stop being just a bookkeeping device, but carry information of their own.

The diagrams in question here are given by simplicial sets. We start with defining a procedure that produces a category out of a simplicial set.

Let $SSet$ be a small category of simplicial sets. For any $\mathcal{S} \in SSet$, $\mathcal{S} = \{\mathcal{S}_n\}_{n\geq 0}$, we define a category S as follows:

- The set of objects in S is $\coprod\limits_{n\geq 0} \mathcal{S}_n$.
- Given two objects $s_m \in \mathcal{S}_m \subseteq S$, $s_n \in \mathcal{S}_n \subseteq S$, $Hom(s_m, s_n)$ is the set of all simplicial operators $\mathcal{S}_m \to \mathcal{S}_n$, that map s_m to s_n.

It is clear that for any $\mathcal{S} \in SSet$, S is a small category. It is also clear that for any morphism $f : \mathcal{S} \to \mathcal{S}'$ in $SSet$ there is a functor $R(f) : S \to S'$, and that this assignment $f \mapsto R(f)$ is functorial. Therefore we have an object $\underline{SSet} := \{SSet, R\} \in \underline{Cat}$.

Proposition 4 *There is a structure of $\ltimes$-monoid on $\underline{SSet}$.*

Proof: We need to define

$$\{\Delta, \delta\} : \underline{SSet} \ltimes \underline{SSet} \to \underline{SSet}, \tag{2}$$

where $\Delta : SSet \ltimes SSet \to SSet$ is a functor, and δ is a natural transformation

$$\begin{tikzcd}
SSet \ltimes SSet \arrow[rr, "\Delta"] \arrow[dr, "R\ltimes R"'] & \arrow[d, phantom, "\xleftarrow{\delta}"] & SSet \arrow[dl, "R"] \\
& Cat &
\end{tikzcd}$$

We start with defining Δ on objects. Let $\{\mathcal{S}, \psi\}$ be an object in $SSet \ltimes SSet$, i.e. $\mathcal{S}$ is a simplicial set, and $\psi : S \to SSet$ is a functor. The set of objects in the category $S \ltimes_\psi \underline{SSet}$ is graded by $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, indeed, objects in S are $\mathbb{Z}_{\geq 0}$-graded by dimension of simplices, and similarly for categories in the image of $R \circ \psi$.

It is easy to see that the category $S \ltimes_\psi \underline{SSet}$ can be obtained from a bisimplicial set $\mathbb{T}$ by considering simplices as objects and bisimplicial operators as morphisms. Let $\mathcal{T} \in SSet$ be the diagonal in $\mathbb{T}$. We set

$$\Delta(\{\mathcal{S}, \psi\}) := \mathcal{T}, \qquad \delta : T \to S \ltimes_\psi \underline{SSet},$$

with δ being given by the diagonal. Thus we have defined (2) on objects.

Let $\{f,\phi\} : \{\mathcal{S},\psi\} \to \{\mathcal{S}',\psi'\}$ be a morphism in $SSet \ltimes SSet$, where $f : \mathcal{S} \to \mathcal{S}'$ is a map of simplicial sets, and ϕ is a natural transformation

$$\begin{tikzcd}
S \arrow[rr, "f"] \arrow[dr, "\psi"'] & \arrow[d, phantom, "\xrightarrow{\phi}"] & S' \arrow[dl, "\psi'"] \\
& SSet &
\end{tikzcd}$$

The pair $\{f,\phi\}$ induces a functor $S \ltimes_\psi \underline{SSet} \to S' \ltimes_{\psi'} \underline{SSet}$ that preserves the $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-grading, and defines a map of bisimplicial sets $\mathbb{T} \to \mathbb{T}'$. Consequently we get a functor $T \to T'$ and a corresponding map of simplicial sets $\mathcal{T} \to \mathcal{T}'$. This completes definition of (2).

Now we turn to associativity of $\{\Delta,\delta\}$. This is rather straightforward, essentially it amounts to associativity of taking diagonals in bisimplicial sets.

It remains to define the unit. There is an obvious $\underline{\mathbf{1}} \to \underline{SSet}$, with $\mathbf{1}$ going to the 1-point simplicial set. ■

Having the $\ltimes$-monoid structure on $\underline{SSet}$, we can talk about $\underline{SSet}$-algebras in Cat. For example, if a category M is closed with respect to taking colimits, we have a canonical structure of a $\underline{SSet}$-algebra on M: given a simplicial diagram in M, take its colimit.

Sometimes, having a $\underline{SSet}$-algebra M we might like to work only with the subcategory of M, consisting of mono-morphisms. It might happen that the action of the entire $\underline{SSet}$ does not preserve the chosen subcategory. In these cases the following definition is useful.

Let M be a category, and let $\mathcal{I}$ be a set of generators of M. For example, if $M = SSet$, $\mathcal{I}$ is the set of standard simplices $\{\Delta[n]\}_{n\geq 0}$. Let $\mathcal{D}_\mathcal{S} : \mathcal{S} \to \mathcal{M}$ be an object in $\underline{SSet}(M)$ (Definition 4).

Definition 5 *Let $I \in \mathcal{I}$.* **An I-point** *in $\mathcal{D}_\mathcal{S}$, is a morphism $I[n] \to \mathcal{D}_\mathcal{S}$ in $\underline{SSet} \ltimes M$, where $I[n]$ is I, considered as a constant diagram over $\Delta[n]$.*

It is clear that for each such $I \in \mathcal{I}$ we obtain a simplicial set $I(\mathcal{D}_\mathcal{S})$ of I-points. If $(F,\phi) : \mathcal{D}_\mathcal{S} \to \mathcal{D}_\mathcal{T}$ is a morphism in $\mathfrak{S}(\mathcal{M})$, it induces a morphism of simplicial sets $(F,\phi)_I : I(\mathcal{D}_\mathcal{S}) \to I(\mathcal{D}_\mathcal{T})$.

Definition 6 *We will say that (F,ϕ) is* **a fibration**, *if F is injective, and for each $I \in \mathcal{I}$ the morphism of simplicial sets $(F,\phi)_I$ is a fibration.*

It is straightforward to check that $\{I[n]\}_{I\in\mathcal{I},n\geq 0}$ is a set of generators for $\underline{SSet}(M)$, and hence we can iterate this definition to get the notion of a fibration in $\underline{SSet}^k(M)$ for any $k \geq 1$.

Let $sset \subset SSet$ be the subcategory consisting of injective morphisms, and let $\underline{sset}$ be the corresponding object in $\underline{Cat}$. Let $\underline{sset}\circ\underline{sset} \subset \underline{sset} \ltimes \underline{sset}$ to be the subcategory of fibrations. Similarly, we define $\underline{sset}^{\circ^k}$ for any $k \geq 1$.

Proposition 5 *The sequence $\{\underline{sset}^{\circ^k}\}_{k\geq 1}$ is stable with respect to the $\ltimes$-monoid structure on $\underline{SSet}$.*

Using this proposition we can regard $\underline{sset}$ itself as a monoid, and hence consider $\underline{sset}$-algebras. Usually it happens that if M is an $\underline{SSet}$-algebra, then the category of mono-morphisms in M is an $\underline{sset}$-algebra.

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